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On the Effect of Spin on the Gravitational Field

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Abstract

Using a solution of Trautman's recently formulated Einstein-Cartan equations, it is shown that if an isolated body is embedded in an expanding universe consisting of a dust of spinning particles, then the local gravitational field of the body is influenced by spin, even when the cosmological constant is neglected.

1. *Introduction*

In an earlier investigation, Einstein & Straus (1945) reached the conclusion that the effect of the expansion of the universe has no influence on the structure of the local gravitational field surrounding an individual star. Later, Pirani (1954) also investigated the same problem but came to a different conclusion by taking into account a non-vanishing cosmological constant, namely, that subject to the O'Brien-Synge (1952, 1960) boundary conditions, the expansion of space *does* influence the structure of the gravitational field surrounding an isolated body. For example, this influence implies that there is a finite maximum radius for the orbit of a particle (near the isolated body) if the particle is not to spiral outwards indefinitely.

Following Trautman's (1972) recent formulation of the Einstein-Cartan equations, there has been considerable interest in the effects of spin on the gravitational field. For example, by considering a Friedman type of universe with 10^{80} aligned neutrons, Trautman (1973) has recently shown that spin and torsion may avert gravitational singularities by giving rise to a minimum radius of 1 cm. Isham, Salam & Strathdee (1973) showed that the effect of interposing *f* gravity on Trautman's model is that spin-aligned hadronic matter will not collapse to densities higher than 10^{17} gm cm^{-3}. This means an increase of the minimum radius from 1 cm to 10^{13} cm. Prasanna (1973a) applied the arguments of Isham, Salam & Strathdee (1973) to finite collapsing objects and found that one can obtain a minimum critical mass for black holes.

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By considering a static perfect fluid of spinning particles in the Einstein–Cartan framework, Prasanna (1973b) has also obtained a space-time metric which, although similar to Schwarzschild's interior solution, no longer represents a homogeneous fluid sphere but one with an equation of state of the form $p = \rho + \rho^{1/2}$ + constant. Moreover, for a solution with an equation of state $p_0/\rho_0 = \frac{1}{3}$, the condition that the spin is real sets in a natural restriction on the mass to radius ratio. It turns out also that the metrics are not of class C^1.

In this paper, we investigate the influence of an expanding universe (with $\Lambda = 0$) consisting of spinning dust on the local gravitational field of a body. Such an effect will be of interest in cosmological observations, since these are made near the surfaces of isolated bodies such as the earth or the moon. Moreover, the situation may throw some light on the problem of formulating a suitable definition of the gravitational field in the interior of a continuous medium. The analysis is carried out in a Robertson–Walker type of universe recently found by Kopczyński (1972) in the framework of Trautman's (1972) Einstein-Cartan theory.

Section 2 contains a brief description of Kopczyński's (1972) solution, and Section 3 deals with the appropriate metric for the local field in the empty region surrounding an isolated body embedded in the universe of Kopczyński (1972). In Section 4 we obtain an approximate solution subject to suitable boundary conditions. Finally, Section 5 contains some discussion and conclusions.

2. *Kopczyński's Solution of the Einstein–Cartan Equations*

Consider a spherically symmetric gravitational field produced by spinning dust of particles. Let $\{\theta^i\}$ $(i = 0, 1, 2, 3)$, be an orthonormal frame of 1-forms given in terms of the spherical polar coordinates (t, r, θ, φ) by

$$\theta^0 = dt, \qquad \theta^1 = e^{\lambda(r,t)}\, dr, \qquad \theta^2 = e^{\mu(r,t)}\, d\theta, \qquad \theta^3 = e^{\mu(r,t)} \sin\theta\, d\varphi \tag{2.1}$$

The Einstein–Cartan field equations are:

$$R_i^{\ j} - \tfrac{1}{2}R\delta_i^{\ j} = -8\pi t_i^{\ j}$$

and

$$Q^i_{\ jn} - \tfrac{1}{2}\delta^i_{\ j}Q^m_{\ mn} - \tfrac{1}{2}\delta^i_{\ n}Q^m_{\ jm} = -8\pi s^i_{\ jn} \tag{2.2}$$

where $t^i_{\ j}$ is the canonical energy-momentum tensor and is related to the usual symmetric energy momentum tensor T^{ij} by

$$T_i^{\ j} = \theta^j{}_\wedge t_i - \tfrac{1}{2}Ds_i^{\ j} \tag{2.3}$$

where s_{ij} is given in terms of the intrinsic angular momentum density tensor $s^n_{\ ij}$ by $s_{ij} = \eta_n s^n_{\ ij}$, where η_k is the dual form of θ^k and $t_i = \eta_j t^j_{\ i}$. The symbol D denotes covariant exterior derivative and $Q^i_{\ jn}$ is the torsion tensor.

In the case of the spinning dust, the symmetric energy-momentum tensor $T^{ij} = \tilde{T}^{ij}\eta$ is taken in the form

$$\tilde{T}^{ij} = \rho u^i u^j \tag{2.4}$$

where ρ is the density of the mass of the dust and $u = u_i\theta^i$ is the 1-form of its velocity. Assuming a classical description of spin, we have

$$s_{ij}{}^l = S_{ij}u^l \qquad \text{and} \qquad S_{ij}u^j = 0 \tag{2.5}$$

where S_{ij} is the tensor of density of spin. Spherical symmetry implies that the only non-vanishing component of S_{ij} is the radial component $S_{23}(t, r)$.

Putting $K = 4\pi S_{23}$, the field equations (2.2) with the continuity equation for spin and an assumption of minimal spin coupling then yield the Robertson-Walker type of solution (Kopczyński, 1972)

$$ds^2 = dt^2 - K^{-3/2}(t)\left\{\frac{1}{1 - Ar^2}\,dr^2 + r^2\,d\Omega^2\right\}$$

$$(d\Omega^2 \equiv d\theta^2 + \sin^2\theta\,d\varphi^2) \tag{2.6}$$

where the density of spin $K(t)$ now reduces to a function of t only and satisfies the differential equation

$$\tfrac{2}{3}K\ddot{K} - \dot{K}^2 + K^4 - AK^{8/3} = 0 \tag{2.7}$$

where A is an arbitrary constant and a dot denotes differentiation with respect to t. The cosmological radius $\mathcal{R}(t)$ of the universe is related to the density of spin by

$$|K(t)|\,\mathcal{R}^3(t) = \begin{cases} |A|^{-3/2}, & \text{if } A \neq 0 \\ 1, & \text{if } A = 0 \end{cases} \tag{2.8}$$

The nature of the solutions of (2.7) is determined mainly by the sign of A. In the particular case where $A = 0$, the solution of (2.7) is

$$|K(t)| = E/(1 + \tfrac{3}{4}E^2t^2) \tag{2.9}$$

where E denotes total 'energy'. (In all cases, the function $|K(t)|$ is bounded and never vanishes.) The density of mass ρ and the density of spin $|K|$ are related by

$$8\pi\rho = E|K| \tag{2.10}$$

and the constant $2/E$ can be interpreted as the amount of spin per unit mass.

3. *The Field Near an Isolated Body*

Consider a spherical empty region (I) of radius a surrounding a non-spinning body of mass m centred at the spatial origin $r = 0$ of an expanding universe of spinning dust given by the metric (2.6). The coordinates (T, R, θ, φ) usually employed in the Schwarzschild exterior metric,

$$ds^2 = \left(1 - \frac{2m}{R}\right)dT^2 - \left(1 - \frac{2m}{R}\right)^{-1} dR^2 - R^2\,d\Omega^2 \tag{3.1}$$

are not co-moving and so (3.1) is not a suitable form for a discussion involving an expanding universe. Accordingly, following Nariai & Tomita (1965), we assume that (3.1) can be transformed into the form:

$$ds^2 = e^{\nu(t,r)}\,dt^2 - e^{\lambda(r,t)}\,dr^2 - K(t)^{-2/3}\,r^2\,d\Omega^2 \tag{3.2}$$

by means of the coordinate transformation of the form

$$T = f(t, r) \tag{3.3}$$

$$R = K(t)^{-1/3} r \tag{3.4}$$

where (t, r, θ, φ) are the same as the co-moving coordinates employed in (2.6).

Now, the transformation (3.3) and (3.4) takes (3.1) to the form

$$\begin{aligned} ds^2 = {} & (1 - 2mK^{1/3}/r)\{\dot{f}^2\,dt^2 + (f')^2\,dr^2 + 2\dot{f}f'\,dt\,dr\} \\ & - (1 - 2mK^{1/3}/r)^{-1}\{\tfrac{1}{9}K^{-8/3}\dot{K}^2 r^2\,dt^2 + K^{-2/3}\,dr^2 - \tfrac{2}{3}K^{-5/3}\dot{K}r\,dt\,dr\} \\ & - K^{-2/3} r^2\,d\Omega^2 \end{aligned}$$

which is identical with (3.2) provided

$$e^{\nu} = (1 - 2mK^{1/3}/r)\dot{f}^2 - \tfrac{1}{9}(1 - 2mK^{1/3}/r)^{-1}K^{-8/3}\dot{K}^2 r^2 \tag{3.5}$$

$$e^{\lambda} = (1 - 2mK^{1/3}/r)^{-1}K^{-2/3} - (1 - 2mK^{1/3}/r)(f')^2 \tag{3.6}$$

and

$$3K^{5/3}(1 - 2mK^{1/3}/r)^2\dot{f}f' + \dot{K}r = 0 \tag{3.7}$$

where the dots and the primes denote, respectively, differentiation with respect to the coordinates t and r.

In what follows, we assume that the solution for the function $f(t, r)$ of the partial differential equation in (3.7), subject to appropriate boundary conditions to be discussed, gives the metric for the empty region (I) via (3.2)-(3.6). The question of suitable boundary conditions is taken up in the next section.

4. *The Boundary Conditions and Solutions*

The problem of formulating suitable junction conditions in general relativity has received considerable attention over the years (O'Brien & Synge, 1952; Synge, 1960; Oppenheimer & Snyder, 1939; Lichnerowicz, 1955; Hoyle & Narlikar, 1964; Nariai, 1965). The difficulty arises from the fact that one must consider not only the prevailing physical conditions but also the smoothness of the coordinate patches used in covering a space-time manifold.

In relation to the above, Lichnerowicz (1955) postulated the existence of admissible coordinates in which the gravitational potentials are functions of class C^1 and piecewise of class C^3, the transformations between systems of admissible coordinates being diffeomorphisms of class C^2 and piecewise of class C^4. It is interesting to note from Prasanna (1973b) that the effect of spin compels one to relax the Lichnerowicz boundary conditions.

O'Brien & Synge (1952) derived the following boundary conditions in a system of coordinates x^i (where $i, j = 0, 1, 2, 3$ and $\alpha, \beta = 1, 2, 3$) relative to which the non-null boundary $\Sigma : x^0 = 0$ is at rest:

$$\left[\overset{(\mathrm{I})}{g_{ij}}\right] = \left[\overset{(\mathrm{II})}{g_{ij}}\right] \tag{4.1}$$

$$\left[\overset{(\mathrm{I})}{g_{\alpha\beta,0}}\right] = \left[\overset{(\mathrm{II})}{g_{\alpha\beta,0}}\right] \tag{4.2}$$

$$\left[\overset{(\mathrm{I})}{T_i^0}\right] = \left[\overset{(\mathrm{II})}{T_i^0}\right] \tag{4.3}$$

$$\left[\overset{(\mathrm{I})}{g_{\alpha i}}\overset{(\mathrm{I})}{T_j^\alpha} - \overset{(\mathrm{I})}{g_{\alpha j}}\overset{(\mathrm{I})}{T_i^\alpha}\right] = \left[\overset{(\mathrm{II})}{g_{\alpha i}}\overset{(\mathrm{II})}{T_j^\alpha} - \overset{(\mathrm{II})}{g_{\alpha j}}\overset{(\mathrm{II})}{T_i^\alpha}\right] \tag{4.3$'$}$$

where the symbol [ ] denotes values taken on the boundary $\Sigma : x^0 = 0$ between the adjacent regions (I) and (II), and (N) above a quantity refers to its value in the region (N), where $N \equiv$ I or II. In the above, the coordinate x^0 can be any one of the coordinates x^i. If the energy-momentum tensor $T_j^{\ i}$ is symmetric, then (4.1) and (4.3) imply (4.3′).

More recently, Nariai (1965) and Nariai & Tomita (1965) have considered the problem of formulating suitable boundary conditions in general relativity. By stipulating that Einstein's field equations constructed from the combined metric (defined as a step-function combination of two metrics) should be delta-singularities free, Nariai (1965) recovered the O'Brien–Synge conditions (4.1)–(4.3) and, in addition, arrived at the following new condition in general relativity:

$$[K^i{}_{j;i}] = \kappa\,[Q_j] \tag{4.4}$$

the details of which are given in the Appendix. The new condition (4.4) reduces to an identity in admissible coordinates (Nariai, 1965). In previous papers (Kofinti, 1972; Kofinti, 1973), we have investigated the new condition (4.4) in some detail and it appears to us that the condition may be non-trivial in non-admissible coordinates.

Returning now to our main discussion, in this paper we shall consider the solution of the differential equation (3.7) subject to the boundary conditions discussed above in (4.1)–(4.3). Corresponding to the coordinate x^0, we specify the boundary by

$$\Sigma : x^0 \equiv r - a = 0$$

since the sphere of Section 3 is of radius **a** (constant). Identifying the universe given by (2.6) with region (II), then from (2.6) and (3.2) the condition (4.1) gives

$$[e^\nu] = 1, \qquad [e^\lambda] = 1/K^{2/3}(1 - Aa^2) \tag{4.5}$$

for all t. In the present case, condition (4.2) is equivalent to

$$\left[\partial \overset{(\mathrm{I})}{g_{\alpha\beta}}/\partial r\right] = \left[\partial \overset{(\mathrm{II})}{g_{\alpha\beta}}/\partial r\right] \tag{4.6}$$

where $\alpha, \beta = 0, 2, 3$ (i.e. $\alpha, \beta \neq 1$). Applying (4.6) to (2.6) and (3.2) we have

$$[\nu'] = 0 \tag{4.7}$$

The boundary condition (4.3) also yields

$$[K^{2/3}]\,[\tfrac{2}{3}K\ddot{K} - \dot{K}^2 - AK^{8/3}] = 0 \tag{4.8}$$

Hence, in view of (2.7), the condition (4.3) is satisfied to at least the order $O[(K^{1/3})^{14}]$. In Isham, Salam & Strathdee (1973), for example, $\mathcal{R}_{\min} \sim 10^{13}$ cm and so, by (2.8), $K_{\max} \sim 10^{-39}$ cm^{-3}, which means (4.3) is reasonably satisfied.

We note also that

$$\left[\partial \overset{(\mathrm{I})}{g}_{11}/\partial r\right] = [\lambda']/\kappa^{2/3}(1 - Aa^2)$$
$$\left[\partial \overset{(\mathrm{II})}{g}_{11}/\partial r\right] = 2Aa/K^{2/3}(1 - Aa^2) \tag{4.9}$$

and so

$$\left[\partial \overset{(\mathrm{I})}{g}_{11}/\partial r\right] \neq \left[\partial \overset{(\mathrm{II})}{g}_{11}/\partial r\right]$$

except when

$$[\lambda'] = 2Aa \tag{4.10}$$

The above means that, in general, the metrics we are employing here are not of class C^1.

Following the procedure outlined in Nariai & Tomita (1965), we find that the solution of the differential equation (3.7) subject to the conditions (4.5) and (4.7) is

$$e^{\nu} = 2mK^{2/3}zr(1 - Aa^3/2mK^{1/3})/a^2F(z, r, t) \tag{4.11}$$

$$e^{\lambda} = azF(z, r, t)/K^{1/3}r \tag{4.12}$$

where

$$F(z, r, t) = H(z, r, t) - aK^{1/3}z(1 - 2mK^{1/3}/r)/2r \tag{4.13}$$

and $z = z(r, t)$ is a parameter determined by the integral relation

$$y^2/x = y(z)\exp\left[(z/2)\int_x^{y(z)} g\,dx/x^2\{1 + (zg/2x)^2\}^{1/2}\right] \tag{4.14}$$

where $x \equiv r/K^{1/3}a$, $y \equiv 1/K^{1/3}$, $g \equiv 1 - 2m/ax$, and $y(z)$ is given on the boundary $x = y$ (i.e. $r = a$) by

$$y(z) = (2mz^2/a)^{1/3}(1 - Aa^2)^{1/3}\,[\{(\sqrt{[1 + 4Aa^2z^2(1 - Aa^2)/27]} + 1)/2\}^{1/3} - \{(\sqrt{[1 + 4Aa^2z^2(1 - Aa^2)/27]} - 1)/2\}^{1/3}] \tag{4.15}$$

In general, the integral in (4.14) cannot be evaluated in terms of elementary functions. Hence, as in Nariai & Tomita (1965), we resort to approximate solutions which are valid in subregions of the (t, r) plane. Assuming for simplicity that $A = 0$ and $a = 1$, we find the approximate solution

$$e^{\nu} = 1 - mK^{1/3}(2/r - r^2 + 3) \tag{4.16}$$

$$e^{\lambda} = (1 + 2mK^{1/3}/r - 2mr^2/K^{1/3})/K^{1/3} \tag{4.17}$$

which is valid in the subregion

$$r \ll d \tag{4.18}$$

where

$$d^2 = \min(1/mK^{1/3} - 1, 1/2mK^{1/3})$$

Finally, in view of the remarks following equation (4.8), the above approximate solution satisfies the boundary condition (4.3) as well.

5. *Discussion and Conclusions*

The approximate solution exhibited in (4.16)-(4.18) suggests that an expanding universe with spin does influence the structure of the local gravitational field surrounding an individual body. Thus, if an astronomical instrument is located near the earth (or the moon), one should, in principle, be able to detect the spin effects due to the rest of the universe assumed to be a continuous distribution of spinning galaxies. Admittedly, such an affect will be small, since, for matter in bulk, spins are likely to cancel out one another with the consequent dominance of the effects due to mass.

In Prasanna (1973b), where static solutions are considered, the observation is made that the fact that the metrics are not of class C^1 may be due to spin not affecting the geometry outside a distribution. However, the solution exhibited in this paper seems to suggest that the above observation may not apply to non-static solutions.

As stated in Kopczyński (1972), in the model (2.6) employed above, the torsion is undefined at the spatial origin $r = 0$. However, by the very nature of the situation considered in this paper, the origin is not regarded as a material point of the universe given by (2.6). Hence the question of singularities at $r = 0$ does not arise in our discussion.

Finally, since the new boundary condition (4.4) was arrived at solely on the basis of the usual Einstein field equations, it will be interesting to see what happens in the framework of the Einstein-Cartan theory.

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Appendix

In the new condition

$$[K^i_{j;i}] = \kappa [Q_j]$$

(; denotes the usual covariant derivative)

$$K_{ij} = E_{ij} - \tfrac{1}{2} g_{ij} E$$
$$E_{ij} = g^{lr} E_{lijr}$$

where E_{lijr} is an expression whose value on the boundary $\Sigma : x^0 = 0$ is given by

$$[E_{lijr}] = -\tfrac{1}{4} [g^{mn}] [\Lambda_{(lr)m} \Lambda_{(ij)n} - \Lambda_{(lj)m} \Lambda_{(ir)n}]$$

where

$$\Lambda_{(ij)l} = \overset{(\mathrm{I})}{\Gamma}_{(ij)l} - \overset{(\mathrm{II})}{\Gamma}_{(ij)l}$$

and

$$\Gamma_{(ij)l} = g_{lr} \Gamma^r_{ij}$$

where Γ^i_{jn} is the usual Christoffel symbol of the second kind and κ is the coupling constant. Q_j is an expression whose value on the boundary is given by

$$[Q_j] = -\tfrac{1}{4} [g^{mn}] [\Lambda_{(lm)n} S^l_j - \Lambda_{(jl)n} S^l_m]$$

where

$$S^l_m = \overset{(\mathrm{I})}{T}{}^l_m - \overset{(\mathrm{II})}{T}{}^l_m$$

The tensor g_{ij} is the combined metric tensor whose definable domain is the combined region (I) and (II) and is introduced by the relation

$$g_{ij} = \overset{(\mathrm{I})}{g}_{ij} \theta_{\mathrm{I}} + \overset{(\mathrm{II})}{g}_{ij} \theta_{\mathrm{II}}$$

where $\theta_{\mathrm{I}} = \theta(x^0)$, $\theta_{\mathrm{II}} = \theta(-x^0)$, and $\theta(x^0)$ is the step-function defined by

$$\theta(x^0) = \begin{cases} 1, & (x^0 > 0) \\ \tfrac{1}{2}, & (x^0 = 0) \\ 0, & (x^0 < 0) \end{cases}$$

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